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Letter to the Editor

On Gautschi's harmonic mean inequality for the gamma function

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Abstract

Let

$$H_n = \inf_{x \in S_n} \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{\Gamma(x_k)} \right)^{-1},$$

where $S_n = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n : \prod_{k=1}^n x_k = 1\}$. Gautschi (SIAM J. Math. Anal. 5 (1974)) showed that $H_2 = 1$ and $H_n < 1$ for all $n \geqslant 9$. In this paper we prove his conjecture that $H_n = 1$ for $n \leqslant 8$.

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1. Introduction

In 1974, Gautschi [7] presented a remarkable mean value inequality for the classical gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt = x^{-1} e^{-\gamma x} \prod_{k=1}^{\infty} \frac{e^{x/k}}{1 + x/k} \qquad (\gamma = \text{Euler's constant}).$$

He proved that for all positive real numbers x the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is greater than or equal to 1, that is,

$$\frac{2}{1/\Gamma(x) + 1/\Gamma(1/x)} \geqslant 1. \tag{1.1}$$

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Refinements of this result can be found in [2,5]. It is natural to look for a generalization of (1.1) to more variables. In a second paper on this subject, Gautschi [8] studied the inequality

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\Gamma(x_k)}\right)^{-1} \geqslant 1 \tag{1.2}$$

under the constraints that $x_k > 0$ $(k=1,\ldots,n)$ and $\prod_{k=1}^{n} x_k = 1$. He asked: for which n is (1.2) true?

Let

$$S_n = \left\{(x_1,\ldots,x_n)\in\mathbb{R}_+^n\colon \prod_{k=1}^{n} x_k = 1\right\} \quad \text{and} \quad H_n = \inf_{x\in S_n}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\Gamma(x_k)}\right)^{-1}.$$

Theorem 1 in [8] states that for all natural numbers n:

$$H_n = 1\Big/\max_{1\leqslant v\leqslant n-1}\ \max_{0\leqslant x\leqslant 1} G_{n,v}(x),$$

where

$$G_{n,v}(x) = \frac{1}{n}\left[\frac{v}{\Gamma(x)} + \frac{n-v}{\Gamma(x^{-v/(n-v)})}\right].$$

Since equality holds in (1.2) if $x_1 = \cdots = x_n = 1$, we conclude that $H_n \leqslant 1$ for all n. Obviously, $H_1 = 1$ and from (1.1) we obtain $H_2 = 1$. Gautschi proved that $H_n < 1$ for $n \geqslant 9$, and based on extensive numerical computations he conjectured that $H_n = 1$ for all $n \leqslant 8$. It is the aim of this paper to establish this conjecture. Then we get the following result:

Theorem. *Let n be a natural number. The harmonic mean inequality*

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\Gamma(x_k)}\right)^{-1} \geqslant 1$$

holds for all positive real numbers $x_1,\ldots,x_n$ *with* $\prod_{k=1}^{n} x_k = 1$ *if and only if* $n \leqslant 8$.

Throughout, we denote by $\psi = \Gamma'/\Gamma$ the logarithmic derivative of the gamma function and by $x_0 = 1.46163\ldots$ the only positive zero of ψ. A collection of the most important properties of ψ and its derivatives is given, for example, in [1]. An interesting survey on inequalities for the gamma and related functions with a detailed list of references can be found in [9].

The numerical values in Section 2 have been calculated by the computer program 'Maple V Release 5.1'.

2. Lemmas

In this section we present several lemmas, which we need to establish our main result. A slightly modified version of the following elementary lemma is given in [6].

Lemma 1. *Let u and v be two real-valued functions, which are continuous on* $(0,1]$ *and differentiable on* $(0,1)$. *Further, let* $u(1)=v(1)=0$ *and* $v'(x)\neq 0$ *for all* $x\in(0,1)$. *If* u'/v' *is strictly increasing on* $(0,1)$, *then* u/v *is also strictly increasing on* $(0,1)$.

The next lemma is proved in [3].

Lemma 2. *The function* $f(x)=x\psi(x)$ *is strictly convex on* $(0,\infty)$. *Further,* f *is strictly decreasing on* $(0,r_0]$ *and strictly increasing on* $[r_0,\infty)$, *where* $r_0=0.21609\ldots$.

A proof for the following monotonicity theorem can be found in [4].

Lemma 3. *Let* $n\geqslant 1$ *be an integer and let* δ *be a real number. The function*

$$g_{\delta,n}(x)=x^{\delta}|\psi^{(n)}(x)|$$

is strictly decreasing on $(0,\infty)$ *if and only if* $\delta\leqslant n$. *And,* $g_{\delta,n}$ *is strictly increasing on* $(0,\infty)$ *if and only if* $\delta\geqslant n+1$.

Lemma 4. *The function*

$$h(x)=\frac{x\psi(x)}{\Gamma(x)} \tag{2.1}$$

is strictly increasing on $[0.79,1.04]$.

Proof. We define for $x\in[0.79,1.04]$:

$$\lambda(x)=x(\psi(x))^2 \quad \text{and} \quad \mu(x)=\frac{x}{\psi(x)}\lambda'(x)=x\psi(x)+2x^2\psi'(x).$$

From Lemmas 2 and 3 we conclude that μ is strictly increasing on $[0.79,1.04]$. Hence,

$$\mu(x)\geqslant\mu(0.79)=2.14\ldots\ .$$

Since ψ is negative on $(0,x_0)$ we obtain $\lambda'(x)<0$. Let $0.79\leqslant r\leqslant x\leqslant s\leqslant 1.04$. Then we get

$$h'(x)\Gamma(x)=\psi(x)+x\psi'(x)-x(\psi(x))^2\geqslant\psi(r)+s\psi'(s)-r(\psi(r))^2=\Delta(r,s), \quad \text{say}.$$

We have

$$\Delta(0.79,0.84)=0.032\ldots \quad \text{and} \quad \Delta(0.84,1.04)=0.094\ldots,$$

which implies $h'(x)>0$. $\square$

Lemma 5. *The function*

$$w(x)=\psi(x)+x\psi'(x)+\frac{\psi(x)[1-\Gamma(x)]}{\log(x)} \tag{2.2}$$

is strictly increasing on $(0,1)$.

Proof. Let

$$a(x) = (x\psi(x))' = \psi(x) + x\psi'(x) \quad \text{and} \quad b(x) = \frac{\psi(x)[1 - \Gamma(x)]}{\log(x)}.$$

Lemma 2 implies that a is increasing on $(0,1)$. Next, we show that b is strictly increasing on $(0,1)$. We apply Lemma 1 with

$$u(x) = \psi(x)[1 - \Gamma(x)] \quad \text{and} \quad v(x) = \log(x).$$

Then we obtain

$$-\frac{u'(x)}{v'(x)} = \alpha(x) + \beta(x), \tag{2.3}$$

where

$$\alpha(x) = x\psi'(x)[\Gamma(x) - 1] \quad \text{and} \quad \beta(x) = x\psi(x)\Gamma'(x).$$

From Lemma 3 we conclude that α is strictly decreasing on $(0,1)$. A simple calculation reveals that $\beta'(x) < 0$ is equivalent to

$$0 < 2x\psi'(x) + x(\psi(x))^2 + \psi(x) = v(x), \quad \text{say}. \tag{2.4}$$

To prove (2.4) for $x \in (0,1)$ we consider two cases.

Case 1: $0 < x \leqslant 0.45$.

Let $c(x) = x\psi(x) + 1$. We have $c(0) = 0$ and $c(0.45) = -0.005\ldots$, so that the convexity of c gives $c(x) < 0$. Further, we have $\psi(x) < 0 < \psi'(x)$. Thus,

$$v(x) = 2x\psi'(x) + \psi(x)c(x) > 0.$$

Case 2: $0.45 \leqslant x < 1$. Let

$$A(x) = 2x^2\psi'(x), \quad B(x) = x\psi(x), \quad \text{and} \quad C(x) = (x\psi(x))^2.$$

The functions A, B, and $-C$ are increasing on $[0.45, 1]$. Hence,

$$xv(x) = A(x) + B(x) + C(x) \geqslant A(0.45) + B(0.45) + C(1) = 1.72\ldots\ .$$

Thus, (2.4) is valid for all $x \in (0,1)$. Therefore, α and β are strictly decreasing on $(0,1)$. From (2.3) we conclude that u'/v' and $b = u/v$ are strictly increasing on $(0,1)$. This implies that $w = a + b$ is also strictly increasing on $(0,1)$ $\square$

Lemma 6. *For all* $x \in (0,1]$ *we have*

$$p(x) = 1 - \frac{1}{\Gamma(1/x)} - \gamma\log(x) \geqslant 0. \tag{2.5}$$

Proof. Let

$$q(x) = \gamma\Gamma(x) + x\psi(x).$$

Differentiation yields

$$-x\Gamma(1/x)p'(x) = q(1/x) \tag{2.6}$$

and

$$q'(x) = \psi(x)[1 + \gamma\Gamma(x)] + x\psi'(x). \tag{2.7}$$

Now, we prove that $q'(x)$ is positive for $x \geqslant 1$.

Case 1: $1 \leqslant x \leqslant x_0$.

Then we have

$$\psi(x)[1 + \gamma\Gamma(x)] \geqslant \psi(1)[1 + \gamma\Gamma(x)] \geqslant \psi(1)(1 + \gamma). \tag{2.8}$$

Applying Lemma 3 we obtain

$$x\psi'(x) \geqslant x_0\psi'(x_0), \tag{2.9}$$

so that (2.7)–(2.9) give

$$q'(x) \geqslant \psi(1)(1 + \gamma) + x_0\psi'(x_0) = 0.50\ldots\ .$$

Case 2: $x_0 < x$.

Since $\psi(x)$ and $\psi'(x)$ are positive, we conclude from (2.7) that $q'(x) > 0$.

Thus, we obtain $q(x) > q(1) = 0$ for $x > 1$. From (2.6) we get that p' is negative on $(0, 1)$. This leads to $p(x) \geqslant p(1) = 0$ for $x \in (0, 1]$. □

Lemma 7. *For all* $x \in (0, 1]$ *we have*

$$\sigma(x) = \frac{8}{7} - \frac{1}{\Gamma(1/x)} - \frac{1}{7\Gamma(x^7)} \geqslant 0. \tag{2.10}$$

Proof. We define

$$\eta(x) = -\frac{1}{\Gamma(1/x)} \quad \text{and} \quad \theta(x) = -\frac{1}{\Gamma(x^7)}.$$

The function η is decreasing on $(0, 1/x_0]$ and increasing on $[1/x_0, 1]$. And, θ is decreasing on $(0, 1]$. To prove (2.10) we distinguish three cases.

Case 1: $0 < x \leqslant 1/x_0$.

Then, $\sigma(x) \geqslant \sigma(1/x_0) = 0.003\ldots\ .$

Case 2: $1/x_0 \leqslant x \leqslant 0.968$.

Let $1/x_0 \leqslant r \leqslant x \leqslant s \leqslant 0.968$. Then

$$\sigma(x) \geqslant \frac{8}{7} - \frac{1}{\Gamma(1/r)} - \frac{1}{7\Gamma(s^7)} = \Lambda(r, s), \quad \text{say.}$$

We have $\Lambda(1/x_0, 0.71) = 0.00007\ldots$. Let

$$r(n) = 0.71 + \frac{n}{102}, \qquad s(n) = 0.71 + \frac{n+1}{102},$$

$$r^*(n) = 0.915 + \frac{n}{300}, \qquad s^*(n) = 0.915 + \frac{n+1}{300}.$$

Then we get

$$\Lambda(r(n), s(n)) > 0 \quad \text{for} \quad n = 0, 1, \ldots, 20, \quad \text{and} \quad \Lambda(r^*(n), s^*(n)) > 0 \quad \text{for} \quad n = 0, 1, \ldots, 15.$$

This leads to $\sigma(x) > 0$ for $x \in [1/x_0, 0.968]$.

Case 3: $0.968 \leqslant x \leqslant 1$.
Differentiation gives

$$x\sigma'(x) = h(x^7) - h(1/x),$$

where h is defined in (2.1). Since

$$0.79\ldots = (0.968)^7 \leqslant x^7 \leqslant 1/x \leqslant 1/0.968 = 1.03\ldots,$$

we conclude from Lemma 4 that $\sigma'(x) \leqslant 0$. This implies $\sigma(x) \geqslant \sigma(1) = 0$. $\square$

Lemma 8. *Let*

$$\tau(x) = 1 - \Gamma(x) + x\log(x)\psi(x). \tag{2.11}$$

There exists a number $\tilde{x} \in (0,1)$ *such that* τ *is negative on* $(0,\tilde{x})$ *and positive on* $(\tilde{x},1)$.

Proof. We have

$$\tau'(x) = \log(x)w(x),$$

where w is defined in (2.2). Applying Lemma 5 we conclude that τ' has at most one zero on $(0,1)$. Since $\tau(1) = 0$, it follows that τ has at most one zero on $(0,1)$, too. We have

$$\lim_{x\to 0}\tau(x) = -\infty, \quad \tau(1) = \tau'(1) = 0, \quad \tau''(1) = \pi^2/6 - \gamma^2 - \gamma = 0.73\ldots,$$

which implies that τ attains positive and negative values on $(0,1)$. Hence, there exists a number $\tilde{x} \in (0,1)$ such that $\tau(x) < 0$ for $x \in (0,\tilde{x})$ and $\tau(x) > 0$ for $x \in (\tilde{x},1)$. $\square$

3. Proof of Gautschi's conjecture

We are now in a position to establish that $H_n = 1$ for all $n \leqslant 8$. According to the results mentioned in Section 1 it is enough to prove that

$$\frac{1}{n}\left[\frac{v}{\Gamma(x)} + \frac{n-v}{\Gamma(x^{-v/(n-v)})}\right] \leqslant 1$$

for $1 \leqslant v \leqslant n-1 \leqslant 7$ and $0 < x \leqslant 1$. We set

$$y = x^{v/(n-v)} \quad \text{and} \quad \rho = n/v - 1.$$

Then it suffices to show that

$$\frac{1}{\Gamma(y^\rho)} + \frac{\rho}{\Gamma(1/y)} \leqslant \rho + 1 \tag{3.1}$$

for $0 < \rho \leqslant 7$ and $0 < y \leqslant 1$.

Clearly, (3.1) is true for $y = 1$. Next, let $y \in (0,1)$ be fixed. We define for $t > 0$:

$$F_y(t) = 1 + \frac{1}{t} - \frac{1}{\Gamma(1/y)} - \frac{1}{t\Gamma(y^t)}$$

and

$$F_y(0) = \lim_{t \to 0} F_y(t) = 1 - \frac{1}{\Gamma(1/y)} - \gamma \log(y).$$

Partial differentiation yields

$$t^2 \Gamma(y^t) \frac{\partial F_y(t)}{\partial t} = \tau(y^t),$$

where τ is defined in (2.11). From Lemma 8 we conclude that there exists a number $\tilde{x} \in (0,1)$ such that

$$\frac{\partial F_y(t)}{\partial t} > 0 \quad \text{for} \quad 0 < t < \log(\tilde{x})/\log(y),$$

and

$$\frac{\partial F_y(t)}{\partial t} < 0 \quad \text{for} \quad t > \log(\tilde{x})/\log(y).$$

This implies

$$\min_{0 \leqslant t \leqslant 7} F_y(t) = \min\{F_y(0), F_y(7)\}.$$

Let p and σ be defined in (2.5) and (2.10), respectively. Applying Lemmas 6 and 7 we obtain $F_y(0) = p(y) \geqslant 0$ and $F_y(7) = \sigma(y) \geqslant 0$. Thus, $F_y(t) \geqslant 0$ for $0 \leqslant t \leqslant 7$. This proves (3.1) for $0 < \rho \leqslant 7$ and $0 < y \leqslant 1$.

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